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# ON DIOPHANTINE APPROXIMATIONS RELATED TO $G$-FUNCTIONS

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I would like to give a short report here about what I talked at the congress "New Aspects of Analytic Number Theory" (解析的整数論の新しい展開) held at November in 2001.

I talked there about the following diagram:

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| | alg. func. (sol. of alg. eq. / FN) | ext. ⟶ | $G$-func. (sol. of lin. diff. eq. / FN + $\alpha$) |
|---|---|---|---|
| rat'l pts | Diophantine approx. | | (this report) |

(FN = the rational functions field over a number field)

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We know two-type Diophantine approximations for algebraic curves. One is on the rational points. See Proposition 1 below and its remark. Another is on the functional fields. See [M], [U] and [Wan].

This report is on "rational points" of $G$-functions.

Contents of this report.

§0 : known results on algebraic curves.

§1 : a very short introduction of $G$-functions and $G$-operators.

§2 : results.

In order to compare the case of algebraic curves and ours, we will recall two known results on rational points of algebraic curves in §0. We will give a very short introduction of $G$-functions and $G$-operators in §1, and we will state our results in the last §.

We notice that this report is short. For more information (our motivations and our view of this topics, close-up comments, a kind of daydreams, and so on), please refer to the latest *SuuriKaisekiKenkyuujyo-Koukyuuroku* (数理解析研究所講究録) [N1].[1]

Throughout this report, $K$ denotes a number field with finite degree $[K:\mathbb{Q}] < \infty$.

[1]The author recommends that the readers refer to the reference.

## §0 Known Results on Algebraic Curves.

First of all, we recall Liouville's inequality.

**Proposition 1 (Liouville's inequality).**
*(We mention about only the special case of genus 0.)*
*Let* $f(x,y) := x - g(y) \in K(x,y)$, $g(y) \in K(y)$, $n := \deg_y g(y)$.
*Put*

$$S_1 := \{g(y) \in K \mid y \in K\} = \{x \in K \mid \exists\, y \in K \ s.t. \ f(x,y) = 0\}.$$

*Let* $t$ *be in* $K$ *with* $\frac{d}{dy} g(t) \neq 0$*: fixed. Put* $a := g(t)$*. Then there exists a positive constant* $c > 0$ *such that*

$$|\alpha - a| > \frac{c}{H(\alpha)^{[K:\mathbb{Q}]/n}} \quad \text{for all } \alpha \in S_1 \text{ with } \alpha \neq a.$$

*(c is independent of* $\alpha$ *.)*
*Here, the symbol* $|\cdots|$ *means the usual absolute valuation, and* $H(\alpha)$ *means the absolute Height of* $\alpha$. □

*Remark.* (See [Se])
It is known that there exist "sharper bounds" for positive genus cases (by using Weil functions and Weil Height) as Roth's theorem (i.e., they don't require a *fixed field* $K$). □
In this report, we will not treat Roth-type, but only Liouville-type.

Next, we recall an estimation on the number of rational points of algebraic curves.

**Proposition 2 (an estimation on the number of rational points of algebraic curves).** *(See* [Se]*)*
*Let* $f(x,y) \in \mathbb{Q}[x,y]$*: an absolutely irreducible polynomial and* $n := \deg_x f(x,y)$*. Put*

$$S_0 := \{x \in \mathbb{Q} \mid \exists\, y \in \mathbb{Q} \ s.t. \ f(x,y) = 0\}.$$

*Then for a closed interval* $[a,b] \subset \mathbb{R}$*, the following holds:*

$$\varlimsup_{B\to\infty} \frac{\log \#\{\zeta \in S_0 \mid H(\zeta) \leq B,\ \zeta \in [a,b]\}}{\log B} \leq \frac{2}{n}. \quad \square$$

We remark the trivial estimation is $\leq 2$.

Moreover we remark :

*Remark (due to Néron, Mumford, Faltings).* (See [Se])
If the curve defined by $f(x,y) = 0$ has the positive genus, then

$$\varlimsup_{B\to\infty} \frac{\log \#\{\zeta \in S_0 \mid H(\zeta) \leq B,\ \zeta \in [a,b]\}}{\log B} = 0. \quad \square$$

The Propositions 1 and 2 are some arithmetical properties of rational points of algebraic curves. We can say that they are about "solutions of algebraic equations $/K(x)$" (i.e., algebraic functions $/K$).
We intend here to extend the topics to the case of "solutions of linear differential equations $/K(x)$ with an arithmetical condition" (i.e., solutions of a $G$-operator $\approx$ $G$-functions).

## §1: a very short Introduction of $G$-functions and $G$-operators.

Roughly speaking,

*a $G$-function is a power series $\in K[[x]]$ solution of a linear differential equation $/K(x)$ such that the Height of its coefficients glows at most geometrically.* (See [A] for the detail definition)

Anyway, a $G$-function is a solution of a linear differential equation $K(x)$ with an arithmetical condition.

Instead of to give the precise definition, we give examples of $G$-functions:

**Examples of $G$-functions.**
(1) Algebraic functions $/K$ (which have power series' expressions).
(2) Polylogarithms. (Of cause, the logarithmic function is also a $G$-function.)
(3) Gauss' hypergeometric series with rational parameters.

We note that algebraic functions $/K$ are $G$-functions. This is the reason why we use the word *extension.* We regard that a $G$-function is an extended notion of an algebraic function /NF in some sense. One can find a reason in so-called Grothendieck's conjecture and some conjectures on $G$-functions. See [Ba]. See also [N1] for elementary comments.

Next, we introduce a term for a linear differential equation itself.
Let $n \in \mathbb{N}$ be given. We consider a linear differential equation:

$$\text{(EQ)} \qquad \frac{d}{dx} m = Am, \quad A \in M_n(K(x))$$

Here we suppose that $m$ is a vector solution.

**Definition.**
We call (EQ) (or $d/dx - A$) a *$G$-operator* if

$$\overline{\lim_{m\to\infty}} \sum_{v\nmid\infty} \frac{1}{m} \max_{i\le m} \log^+ \left| \frac{1}{i!}\left(\frac{d}{dx} + {}^tA\right)^i I \right|_v < \infty,$$

Here, the summation $\sum_{v\nmid\infty}$ means $v$ runs every normalized non-Archimedean valuation of $K$, and $I$ is the identity matrix, the symbol $|\cdots|_v$ is so-called the Gauss norm at $v$.

One might think that this expression is not so intuitive. Anyway, it is known that the notion of $G$-functions and the notion of $G$-operators are *almost* equivalent.

$$G\text{-function} \underset{\text{almost}}{\approx} G\text{-operator}$$

That is, under some conditions, relevant solutions of a $G$-operator are $G$-functions, and vice versa. See [C], [A], [N2].

Anyway, a $G$-operator means a linear differential equation with this arithmetical condition.

Now, our target is an analytic solution of a $G$-operator.[2]

[2]The definition of a $G$-function is *local*, that is, it is a local solution with a power series expression. The conditions of a $G$-function are very local, and thus a $G$-function is a local notion. The other hand, a $G$-operator is a global notion. The condition of a $G$-operator is global. Although solutions of a $G$-operator are also local objects, we prefer local objects in a global definition = (analytic solutions of a $G$-operator) to just local objects (= $G$-functions). (Of cause they are almost equivalent ...)

## §2: Results.

Let $d(x) \in \mathbb{Z}[x]$ (be a polynomial on rational integers): the common denominator of components of $A$ in (EQ).

**Theorem.**
*Let $D$ be a closed disk $\subset \mathbb{C}$ centered $\zeta_0 \in K$ (given) with the radius $< 1/2$ (: given)*
*For a vector solution of* (EQ)*: $m = {}^t(f_1 \dots f_n)$ (: given)*
*we suppose that*
*(0)* (EQ) *is a $G$-operator, and $n \geq 2$*
*(1) $m$ is analytic on $D$ and $f_1, \dots, f_n$ are linearly independent over $\mathbb{C}(x)$,*
*(2) There exist no solutions of $d(x) = 0$ on $D$.*
*Put*

$$S_K := \{\zeta \in D \cap K \mid \exists\ \kappa_\zeta \in \mathbb{C}, \neq 0 \ s.t.\ \kappa_\zeta m(\zeta) \in K^n\}$$

*Then we have*
*(a) If $\zeta_0 \in S_K$, then for any small positive $\epsilon > 0$, there exists a finite constant $c < \infty$ such that*

$$|\zeta_0 - \zeta| \geq \frac{1}{H(\zeta)^{[K:\mathbb{Q}](\frac{1}{n}+\epsilon)}} \quad \textit{for all } \zeta \in S_K \textit{ with } H(\zeta) \geq c.$$

*Here, $c$ is independent of $\zeta$, ($c$ depends on $\zeta_0$, $\epsilon$, $A$,) and effective.*

*And we also have*
*(b)*

$$\varlimsup_{B\to\infty} \frac{\log \#\{\zeta \in S_K \mid H(\zeta) \leq B\}}{\log B} \leq \frac{4}{n}[K:\mathbb{Q}]. \quad \square$$

Here the trivial estimation is $\leq 2[K:\mathbb{Q}]$ by Schanuel's estimation.

We recall Proposition 2. The estimation of the case of algebraic curves is $2/n$, and our estimation is $4/n$.[3]

We remark that we had reported an estimation of type (b) in the last century:

*Remark.* [N3] (1999)

$$\varlimsup_{B\to\infty} \frac{\log \#\{\zeta \in S_K \mid H(\zeta) \leq B\}}{\log B} \leq 1 + \frac{35}{6n}$$

if $K = \mathbb{Q}$ for $n \geq 6$. $\square$
So, our estimation in Theorem is sharper.

As corollaries of (a) and (b):

**Corollary.**
*Under the assumptions in Theorem, we assume that $f_1, \dots, f_n$ are homogeneously algebraically independent over $\mathbb{C}(x)$, and assume that they are $G$-functions.*
*Then we have*
*(c) If $\zeta_0 \in S_K$, then for any small positive $\epsilon > 0$, there exists a finite constant $c < \infty$ such that*

$$|\zeta_0 - \zeta| \geq \frac{1}{H(\zeta)^\epsilon} \quad \textit{for all } \zeta \in S_K \textit{ with } H(\zeta) \geq c.$$

*Here $c$ is effective. ($c$ depends on $\zeta_0, \epsilon, A$.)*
*And also,*
*(d)*

$$\varlimsup_{B\to\infty} \frac{\log \#\{\zeta \in S_K \mid H(\zeta) \leq B\}}{\log B} = 0. \quad \square$$

[3] Of cause the symbol $n$ denotes different meanings (degree | rank).

If we pay attention to only the expressions of (c) and (d), one finds that they are similar to the case of algebraic curves with positive genus.

We give some examples for our results.

**Example 1.**
$s \geq 2$: a natural number. We consider the curve $x^s + y^s = 1$.
Put $\zeta_0 = 0$, $D$: a closed disk centered 0 with the radius 1/3,

$$S_K := \{\zeta \in D \cap K \mid y = \sqrt[s]{1-\zeta^s} \in K\}.$$

Then we have for any small $\epsilon > 0$, there exists an effective constant $c < \infty$ such that

$$|\zeta_0 - \zeta| \geq \frac{1}{H(\zeta)^{[K:\mathbb{Q}](\frac{1}{2}+\epsilon)}} \quad \text{for all } \zeta \in S \text{ with } H(\zeta) \geq c. \quad \square$$

**Example 2.** (See also (Wo).)
Let

$$y(x) := \frac{1}{\pi}\int_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}}, \quad w(x) := \frac{1}{\pi}\int_0^1 \frac{(1-xt)dt}{\sqrt{t(1-t)(1-xt)}},$$
$$D := \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/3\}.$$

Put

$$S_K := \{\zeta \in K \cap D \mid w(\zeta) \neq 0 \text{ and } y(\zeta)/w(\zeta) \in K\}.$$

Then (c) and (d) hold. $\square$

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